Mechanics and Relativity Notes

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Part 1: Single Particle Physics

1 Laws of Motion

1.1 Reference frames

Definition 1. *A **reference frame** consists of a choice of origin and a set of axes. With these, we can determine the position, velocity and acceleration of a particle.*

Definition 2. ***position vector**:* $\vec{x}(t) = \vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$

Definition 3. ***velocity vector**:* $\vec{v}(t) = \frac{d\vec{x}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{x}\hat{z}$

Definition 4. ***acceleration vector**:* $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$

Definition 5. $\vec{a} = \dot{\vec{v}} = \ddot{\vec{x}}$

1.2 Newton's laws of motion

1. **1st law:** The ojects left alone maintain their velocity.
2. **2nd law:** When a force is enacted on an object,
$$\vec{F}_{\text{net}} = m_I \vec{a} = \frac{d\vec{p}}{dt} \tag{1}$$
where $\vec{p}$ is **momentum**.
Equation (1) is called the **equation of motion**.
3. **3rd law:** For every action there is an equal and opposite reaction.

Definition 6. *An **inertial frame** is a frame that is moving at a constant velocity.*

Remark 1. *Newton's laws only apply to inertial frames.*

Definition 7. *The **Principle of Relativity** states that the laws of physics are the same in all inertial frames. In other words, inertial frames are equivalent, so we can choose the one that is most convenient for us when we want to analyse a situation.*

1.3 Galilean transformations

1. **Translations:** $\vec{x}' = \vec{x} + \vec{a}$ Meaning: there is no special point in space.
2. **Rotations:** $\vec{x}' = R\vec{x}$ Meaning: there is no special direction
3. **Galilean boost:** $\vec{x}' = \vec{x} + \vec{v}t$ Meaning: there is no special velocity (well, except for speed of light)
4. **Time translation:** $t' = t + c$ Meaning: there is no special moment.

2 Forces

2.1 Conservation of momentum

Remark 2. *If $\vec{F}_{net} = 0$, then from Newton's 2nd law, it holds that $\frac{d\vec{p}}{dt} = 0$. This means the momentum is constant in time. It also holds that the acceleration is zero, which means the velocity is constant.*

Method 1. *We can solve for the position vector by using*

$$\vec{F}_{net} = m\frac{d^2\vec{r}}{dt^2} \tag{2}$$

To solve for $\vec{r}$ we need to know $\vec{F}(t)$. However, usually we know $\vec{F}(\vec{x})$.

2.2 Motion in 1D

Definition 8. ***Conservative forces*** *are forces that can be expressed as*

$$F(x) = -\frac{d}{dx}V(x) \tag{3}$$

where $V(x)$ is the potential energy. The work done by these forces only depends on the initial and final state of the particle, not on the path taken.

Remark 3. *If $\vec{F}$ is conservative, the energy can be expressed as*

$$E = T + V = \frac{1}{2}mv^2 + V(x) = \frac{1}{2}m\dot{x}^2 + V(x) \tag{4}$$

where T is the kinetic energy. Using the equation of motion (given in Newton's second law), we see $\dot{E} = 0$, so energy is conserved and does not change with time.

Remark 4. *The potential energy $V(x)$ is defined up to an additive constant: V and $V + V_0$ give the same force when we take the derivative.*

SOME CONSERVATIVE FORCES

Force	F	$V(x)$	meaning of x
Ideal spring force	$-kx\hat{x}$	$\frac{1}{2}kx^2$	displacement from equilibrium
Gravitational force on Earth	$-mg\hat{x}$	mgx	distance above "0"
Gravitational force	$-\frac{GMm}{x^2}\hat{x}$	$-\frac{GMm}{x}$	distance between particles
Electrostatic force	$-\frac{kq_1q_2}{x^2}\hat{x}$	$-\frac{kq_1q_2}{x}$	distance between particles

Remark 5. ***Non-conservative forces*** *dissipate mechanical energy by converting it into other forms, typically heat.* ***Friction*** *and* ***air resistance*** *are non-conservative forces. Friction has no potential energy function V and air resistance depends on the path the particle takes.*

2.3 Use of Energy

Method 2. *A more general way to solve for $x(t)$ is to notice that $F = -\frac{dV}{dx}$ gives us the ordinary differential equation $m\ddot{x} = -\frac{dV}{dx}$ where V is some function of x. The general solution to an ODE of the form $a\ddot{y}+by = A$ is $y = Kcos(\sqrt{\frac{b}{a}}t)+\frac{A}{b}$. This method is very similar to method 1, with the extra step of setting $F = -\frac{dV}{dx}$.*

Example 1. *For a harmonic oscillator, we get the following equations:*

- $\vec{F} = -k(x - x_0)\hat{x}$ *where k is the spring constant and x_0 is the position when the spring is not stretched. We usually set this to 0 for convenience. We are allowed to do this because different inertial frames are equivalent. Setting $x_0 = 0$ defines x_0 to be the origin of the axes for our reference frame.*
- *the amplitude $= x_{max}$. At the beginning, to set this all in motion, we pull the mass to x_{max} and let it go.*
- $x(t) = Acos(\omega t) + Bsin(\omega t)$ *where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency*
- *the period $T = \frac{2\lambda}{\omega}$*
- $t_f - t_i = \sqrt{\frac{m}{2}}\int_{x_i}^{x_f} \frac{1}{\sqrt{E-V(x)}}dx = \sqrt{\frac{m}{2}}\int_{x_i}^{x_f} \frac{1}{\sqrt{\frac{1}{2}kx_{max}^2-kx^2}}dx$

2.4 Potential energy graphs

Method 3. *To find out more about a particle's behaviour, we can plot the x, V-graph. Then, given a constant $E = \frac{1}{2}m\dot{x}^2 + V(x)$, we see V is confined to stay below E. This means the particle cannot move from one place to another if that requires V to be greater than E at any point in its path.*
Another thing worthy of note is that, since $E = T + V = \frac{1}{2}m\dot{x}^2 + V$, if we know E and V, the difference between the two is equal to $T = \frac{1}{2}m\dot{x}^2$. So when $E \neq V$, $T \neq 0$ so the particle has a nonzero velocity.
The following situations can arise:

- *The particle is at the a critical point where $\dot{V} = 0$. If $\ddot{V} < 0$ this is a stable point and the particle oscillates around the critical point.*
- *If the critical point has $\ddot{V} < 0$ it is an unstable point and the particle can move very differently depending on the direction of the velocity and the shape of the graph.*

3 Fields

3.1 Motion in 3D

Remark 6. *In 3D, forces are expressed as follows:*

$$\vec{F}(\vec{r}) = \vec{F}(x, y, z) = m\frac{d^2\vec{r}}{dt^2} \tag{5}$$

Definition 9. *In 3D, a force field is called* ***conservative*** *if* $\vec{F}(\vec{x}) = -\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$.

Energy conservation in 3D

If we know V, then it follows that F is conservative and $\dot{E} = 0$. This in turn means $E = T + V = \frac{1}{2}m\vec{v}^2 + V$ where $\vec{v}^2 = v_x^2 + v_y^2 + v_z^2$.

Work in 3D

$$T_f - T_i = -(V_f - V_i) = \int_C \vec{F} \cdot d\vec{x} \tag{6}$$

Remark 7. *If* $\vec{F} \cdot d\vec{x} = 0$ *(which implies* $\vec{F} \perp d\vec{x}$*) then* $W = 0$*. This in turn implies that* $T_f = T_i$ *so the velocity is constant.*

Remark 8. $\int \vec{F} \cdot d\vec{x}$ *is path-independent if* F *is conservative.*

Definition 10. *The* ***angular momentum*** $\vec{L} = \vec{r} \times \vec{p} = (\vec{r} \times \vec{v})m$ *where* $\vec{r}$ *is the position vector and* $\vec{p}$ *is the momentum vector* $m\vec{v}$*.*

Remark 9. *The total angular momentum of a collection of masses* $\vec{L}^{TOT} = \sum L_i$*.*

Remark 10. *The* ***right hand rule*** *works as follows: curl your fingers in the direction of the velocity. Then your thumb points in the direction of the angular momentum.*

Definition 11. *The* ***torque*** $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$*.*

Remark 11. *To maximize the torque, maximize* $\vec{r}$*.*

Remark 12. *Earlier we saw that* $\frac{dp}{dt} = F$*, so the derivative of the momentum is equal to the net force on the particle. We now also have*

$$\frac{dL}{dt} = \tau \tag{7}$$

so the time derivative of the angular momentum gives the torque.

Definition 12. *A **central force** is a force $\vec{F}(\vec{r}) = F(r)\hat{r}$ that only depends on r. The potential energy is then $V = V(r)$. Angular momentum is conserved for central forces.*

Definition 13. *A **gravitational field** is a condition in space created by a mass M which exerts a force on other masses m*

$$\vec{F_g} = -m\vec{\nabla}\Phi_M \tag{8}$$

where Φ_M is the gravitational field caused by M.

Definition 14. *The **gravitational potential** is*

$$V_g = m\Phi_M \tag{9}$$

Remark 13. *At a point P, the gravitational field is equal to*

$$\Phi_M = -\frac{GM}{|\vec{r} - \vec{r_i}|} \tag{10}$$

where G is the gravitational constant, $\vec{r}$ is the vector position of P and $\vec{r}_i$ is the vector position of M.

Definition 15. *The **Principle of Superposition** states that $\Phi(r)$ of many masses is equal to $\sum_{i=1}^{N} \Phi_i = -G\sum_{i=1}^{N} \frac{M_i}{|\vec{r}-\vec{r_i}}$ where $\vec{r}$ is the position vector of the particle being affected.*

Definition 16. *the **Shell Theorem** states:*

1. *For outside: a sphericallly symmetric object affects other objects gravitationally as if all of its mass was concentrated at its center.*
2. *For inside: the gravitational potential is uniform (constant).*

Remark 14. *There are three notable consequences of the Shell Theorem:*

- *Given: a spherical shell (so a hollow sphere), two masses with m_1 inside the shell and m_2 outside of it: the distance r_1 of m_1 from the center of the shell and the distance r_2 between m_2 and the center of the shell.*
  Then the gravitational force on m_1 due to the shell is 0: because of part of 2 of the Shell Theorem, the gravitational field is constant inside the sphere, so its gradient is 0.
  The gravitational force on m_2 is equal to $-\frac{GMm}{r_2^2}\hat{r}_2$.
- *Given: a spherical solid ball; two masses m_1 and m_2; distance r_1 of m_1 from the centre of the sphere and distance r_2 between the two masses.*
  The force due to the mass outside cancels out because of the second part of the Shell Theorem. So only the enclosed mass contributes.
- *For Earth: R_E is the radius of the Earth and z is the height of a mass above the Earth's surface. Then, thanks to part 1 of the Shell Theorem, $V_g(R_E + z) = -\frac{GM_E m}{R_E + z}$.*

Definition 17. *The* ***Equivalence Principle*** *states that gravity interacts with masses in the same way other forces do. In formal terms, by Newton's second law:*

$$\vec{F} = m_I \vec{a} = -m_G \vec{\nabla} \Phi \rightarrow m_I = m_G \tag{11}$$

4 Dimensional Analysis

The goal of dimensional analysis is to check the consistency of equations and to predict the relationship between physical quantities.

Definition 18. *A **dimension** is the physical nature of a quantity, independent of its units. There are 5 fundamental dimensions. All other dimensions are obtained by taking a product of these:*

- *T: time*
- *L: length*
- *M: mass*
- *θ: temperature*
- *Q: charge*

Some basic examples of dimensions are:

- *velocity has dimension $\frac{L}{T}$*
- *acceleration has dimension $\frac{L}{T^2}$*
- *the gravitational constant G has dimension $\frac{L^3}{MT^2}$*
- *force has dimension $\frac{ML}{T^2}$*
- *momentum has dimension $\frac{ML}{T}$*
- *energy has dimension $\frac{ML^2}{T^2}$*

Remark 15. *Quantities can only be added and subtracted if they have the same dimension.*

Remark 16. *Both sides of an equation must have the same dimension.*

Method 4. ***How to use dimensional analysis to solve for Y***

1. *Find $[Y]$*
2. *Choose the relevant variables: X_1, X_2, X_3*
3. *Assume $Y = CX_1^\alpha X_2^\beta X_3^\gamma$ where C is a dimensionless constant*
4. *Find the powers by requiring dimensional consistensy: $[Y] = [X_1]^\alpha [X_2]^\beta [X_3]\gamma$*
5. *Finally, $Y = CX_1^\alpha X_2^\beta X_3^\gamma$*

Remark 17. *If dimensional analysis fails, there is a mistake; if it works, it is not necessarily right.*

5 Planetary Motion

Topic: solutions to a system of two objects ($M >> m$) interacting with a central force.
For motion in 3D, it holds that $m\ddot{\vec{r}} = -\vec{\nabla}V(r) = -\frac{dV}{dr}\hat{r}$.

5.1 Reduction to 2D

The angular momentum $\vec{L}$ is conserved with central forces. $\vec{L} = \vec{r} \times \vec{p} = (\vec{r} \times \vec{v})m = (\vec{r} \times \dot{\vec{r}})m$, so the angular momentum is parallel to the cross product of the position vector and the velocity vector. This means $\vec{L}$ is perpendicular to both $\vec{r}$ and $\dot{\vec{r}}$. These two vectors span a plane, with the angular momentum parallel to the normal vector of the plane. Since the angular momentum is conserved, the plane spanned by $\vec{r}$ and $\dot{\vec{r}}$ stays the same. This means the movement of the small mass is confined to this 2-dimensional plane.

5.2 Polar coordinates

For polar coordinates, we use the basis vectors $\hat{r}$ and $\hat{\theta}$. These are not constant, unlike $\hat{i}$ and $\hat{j}$. $\hat{r}$ points in the direction of increasing r and $\hat{\theta}$ points in the direction of increasing θ.
Expressing relevant variables in polar coordinates (NOTE: the precise forms aren't very important to remember. Main one to know: $\omega = \dot{\theta}$, since that's the definition):

- $\mathbf{x} = r * cos(\theta)$
- $\mathbf{y} = r * sin(\theta)$
- **velocity** $\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r(\dot{\theta}\hat{\theta})$
- **acceleration** $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$
- **angular speed** $\omega = \dot{\theta} = \frac{l}{r^2}$
- **angular momentum** $\vec{L} = mr^2\dot{\theta}\hat{z} = ml\hat{z}$
- **angular momentum per unit mass** $l = r^2\dot{\theta}$

5.3 Reduction to 1D

Using the above, we can rewrite the equation of motion to be $m\ddot{r} = mr(\frac{l}{r})^2 - \frac{dV}{dr}$, which is in terms of r only.
Earlier, we saw that in 1D, it holds that $m\ddot{x} = -\frac{dV}{dx}$. We have a similar expression for central forces in terms of r: $m\ddot{r} = -\frac{dV_{\text{eff}}}{dr}$, where $V_{\text{eff}} = \frac{ml^2}{2r^2} - \frac{GMm}{r}$ with perhaps some constant.
The formula for energy in polar coordinates is $E = \frac{1}{2}m\dot{\vec{r}} - \frac{GMm}{r} = \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} - \frac{GMm}{r}$.

Like earlier, we can draw an effective potential energy diagram. The main takeaway from this diagram is that, for a particle to have a closed orbit, its energy must be negative.

5.4 Solving the orbit equation

By using u-substitution (using $r(\theta)$ instead of $r(t)$, and later setting $u = \frac{1}{r}$), we end up with the inhomogeneous second-order differential equation
$\frac{d^2u}{d\theta^2} + u = \frac{GM}{l^2}$.
The general solution to such an equation is $u = Bcos(\theta) + Csin(\theta) + \frac{GM}{l^2}$. Since the sine function is a shifted cosine function, we can rewrite this as $Acos(\theta - \theta_0) + \frac{GM}{l^2}$, where A and θ_0 depend on the initial conditions. Using $u = \frac{1}{r}$ finally gives us:

$$r = \frac{l^2}{GM(\frac{Al^2}{GM}cos(\theta - \theta_0) + 1} \tag{12}$$

For convenience, we choose θ_0 in such a way that the perigee (the point where the particle is closest to the origin) occurs at $\theta = 0$. r is smallest when $cos(\theta - \theta_0) = 1$, so when $\theta - \theta_0 = 0$, which means we need to choose $\theta_0 = 0$.

Finally, we see that the equation for r describes a conic section. We can rewrite it as $r = \frac{r_0}{ecos(\theta)+1}$, where $r_0 = \frac{l^2}{GM}$ and e is the eccentricity $\frac{Al^2}{GM}$.

6 Planetary Motion and Kepler's Laws

6.1 Classification of orbit in terms of E and e

$$E = \frac{G^2M^2m}{2l^2}(e^2 - 1) \tag{13}$$

- $e = \frac{Al^2}{GM}$
- $l = \sqrt{GMr_0}$

All of the following information can be further conceptualised using a graph of V_{eff} against r.

6.1.1 Negative e

For this course, e will always be positive. This is because $r = \frac{r_0}{ecos(\theta)+1}$, so the negative e can be "absorbed" by a phase shift in the cosine function. This would mean we choose a different θ_0.

6.1.2 e = 0

- Circular orbit with source of gravity at the center
- $E = -\frac{G^2M^2m}{2l^2} = E_{\min}$

6.1.3 0 < e < 1

- Planets in our solar system
- Elliptic orbit with the source of gravity at one of the focii; this point is still considered the origin.
- $E_{\min} < E < 0$
- $\frac{r_0}{1+e} \leq r \leq \frac{r_0}{1-e}$
- Equation of orbit:

$$\frac{(x + \frac{r_0e}{1-e^2})^2}{(\frac{r_0}{1-e^2})^2} + \frac{y^2}{(\frac{r_0}{\sqrt{1-e^2}})^2} = 1 \tag{14}$$

- semi-major axis $a = \frac{r_0}{1-e^2}$
- semi-minor axis $b = \frac{r_0}{\sqrt{1-e^2}}$
- coordinates of center $(-\frac{r_0e}{1-e^2}, 0)$

Important note: The exam will not necessarily use the same letters for the semi-minor and semi-major axes as I have done.

6.1.4 e = 1

- Parabolic trajectory
- $E = 0$. The energy is just barely enough to provide the particle with the minimum escape velocity.
- $r_{\min} = \frac{r_0}{1+e}$

6.1.5 e > 1

- Orbit forms a hyperbola with the center of gravity at one of the focii
- $E > 0$
- Equation of orbit:

$$r(\theta) = \frac{(x - \frac{r_0 e}{e^2-1})^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{15}$$

  where $a = \frac{r_0}{e^2-1}$ and $b = \frac{r_0}{\sqrt{e^2-1}}$
- Asymptotes happen at angles $cos(\theta) = -\frac{1}{e}$ since that is where r goes to infinity.
- The vertex has $r_{\min}$. This is found at $r = \frac{r_0}{1+e}$. There, $\theta = 0$.

6.2 Kepler's Laws

1. Each planet moves in an ellipse with the Sun at one focus.
2. The planets sweep out equal areas in equal times: $\frac{dA}{dt}$ is constant
3. A planets orbital period and its semi-major axis are related: $T^2 \propto a^3$

- $\frac{dA}{dt} = \frac{L}{2m} = \frac{l}{2}$
- $l = \sqrt{GMr_0}$
- $b = \sqrt{ar_0}$
- $T = \frac{2\pi ab}{l} = \frac{2\pi}{\sqrt{GM}} a^{\frac{3}{2}}$

Part 2: Many Particle Physics

7 System of Particles

7.1 Motion of n-particles

An **n-particle** is a group of n particles. For the ith particle, $\vec{p_i} = m_i v_i$ and $\vec{F_i} = \frac{d\vec{p_i}}{dt}$.
A **free particle** is one that has a constant momentum. That means that either the net force acting on the particle is zero or that the particle is travelling at a constant velocity.

$\vec{F_i} = \vec{F}^{\text{ext}} + \sum_{i \neq j} \vec{F}_{ij}$.
We assume for now that these external forces are conservative, so $\vec{F}_i^{\text{ext}} = -\vec{\nabla}_i V_i(\vec{x}_i)$. $\vec{F}^{ext}$ is the sum of all external forces.

The internal forces come in pairs due to Newton's 3rd law: $\vec{F}_{ij} = -\vec{F}_{ji}$, where the first is the fore on i due to j and the second is the force on j due to i.

An **isolated system** has only internal forces. There is no net external force and no energy is exchanged with its surroundings.

For $\vec{F}^{\text{ext}} \neq 0$: $\sum \vec{F}_i^{\text{ext}} \neq 0 \rightarrow \frac{d\vec{p}^{tot}}{dt} = \sum \vec{F}_i^{ext}$.

7.2 Center of Mass

$$\vec{R}_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i \tag{16}$$

All the momenta add up as if all the mass were concentrated at the CM, moving with $\dot{R}_{\text{CM}}$: $\vec{p}^{tot} = M\dot{\vec{R}}_{\text{CM}}$.
If $\vec{F}^{\text{CM}} \neq 0$, then $\dot{\vec{p}}^{tot} = M\ddot{\vec{R}}_{\text{CM}}$. The acceleration of the CM is thus determined only by external forces.

7.3 Coupled Systems

In a **coupled system**, two objects interact with each other: the motion of one is influenced by the other and vice versa.
When dealing with multiple particles, it is often more convenient to work in the CM frame. We then find new position vectors $\vec{y}_i = \vec{R}_{\text{CM}} - \vec{r}_i$. Then $\dot{\vec{y}}_i = \dot{\vec{R}}_{\text{CM}} - \vec{v}_i$.

7.3.1 Turning a two-body problem into a one-body problem

If we have an isolated system of two particles, one with mass m_1 and position vector $\vec{r}_1$ and one with mass m_2 and position vector $\vec{r}_2$, we can reduce the problem to a situation with only one fictitious particle u. We do this by setting $r_u = \vec{r}_1 - \vec{r}_2$ and setting $m_u = \mu = \frac{m_1 m_2}{m_1 + m_2}$. Then $\mu\ddot{\vec{r}} = \vec{F}_{12} = -\vec{\nabla}V$.

If $m_2 \gg m_1$, μ goes to m_1 and $\vec{r}_2$ goes to $\vec{R}_{CM}$, which lies in the center of the second mass.

7.4 Rotation and Energy

$\vec{L}^{\text{tot}} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i$ about some origin.

$\vec{\tau} = \frac{d\vec{L}^{\text{tot}}}{dt} = \sum \vec{r}_i \times \vec{F}_i^{ext}$. This means the total torque is not affected by internal forces.

To see why this is so, we derive it (here, $\vec{F}_{ij}$ is the force on i due to j. It is equal to $-\vec{F}_{ji}$.

$\vec{\tau} = \sum \vec{r}_i \times (\vec{F}_i^{\text{ext}} + \sum_{i \neq j} \vec{F}_{ij}) = \sum \vec{r}_i \times \vec{F}_i^{\text{ext}} + \sum \vec{r}_i \times \sum_{i \neq j} \vec{F}_{ij}$

Looking more closely at the second part of the sum, we see:

$\sum \vec{r}_i \times \sum_{i \neq j} \vec{F}_{ij} = \sum_i \sum_{i \neq j} \vec{r}_i \times \vec{F}_{ij} = \sum_i \sum_{i<j} \vec{r}_i \times F_{ij} + \vec{r}_j \times \vec{F}_{ji}$.

In this last term, we set $i < j$ in the inner sum to avoid double counting. However, we do still count the effects of each force twice. Now they're just inside the inner sum.

Next, this is equal to:

$\sum_i \sum_{i<j} \vec{r}_i \times \vec{F}_{ij} - \vec{r}_j \times \vec{F}_{ij} = \sum_i \sum_{i<j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$.

But $(\vec{r}_i - \vec{r}_j)$ is parallel to $\vec{F}_{ij}$, so this cross product is equal to zero.

The kinetic energy $T = \frac{1}{2} \sum m_i \vec{v}_i \cdot \vec{v}_i$.

In the frame of the center of mass, this becomes $T = \frac{1}{2} M \dot{\vec{R}}_{\text{CM}}^2 + \frac{1}{2} \sum m_i \dot{\vec{y}}_i^2$, where the first is the kinetic energy of the CM and the second is the kinetic energy of the particles moving with respect to the CM point.

For a system that is both spinning and translating, we can break the kinetic energy up into pure translation and pure spinning. In fact, the first part of the equation we had for T above is the part that comes from translation and the second part comes from spinning. To find $\dot{\vec{R}}_{\text{CM}}^2$ and $\dot{\vec{y}}_i^2$, we can use that $v = d\omega$ where d is the distance from the axis of rotation.

When the forces in a system are conservative, the energy is $E = T + \sum_i V_i(x_i) + \sum_{i<j} V_{ij}(|x_i - x_j|)$. The exact solution to this system is impossible to find unless $\vec{F}$ is very special.

8 Collisions, Intro to Rigid Body Motion

8.1 Collisions

- **Elastic collisions** conserve E and $\vec{p}$
- **Inelastic collisions** conserve $\vec{p}$, but mechanical energy is transformed into thermal energy.

Since for any collision, $\frac{d\vec{p}^{tot}}{dt} = 0$, the total momentum of the two particles does not change. Since $\frac{d\vec{p}}{dt} = \vec{F}$, the following equation holds:

$$\vec{p}_a(t_f) - \vec{p}_a(t_i) = \int_{t_i}^{t_f} \vec{F}_{ab} dt = I = -\int_{t_i}^{t_f} \vec{F}_{ba} = -(\vec{p}_b(t_f) - \vec{p}_b(t_i)) \qquad (17)$$

where I is the **impulse**. From the perspective of impulse, gravitational interaction is also a collision.

Solving for position in a collision
CoE gives us $\frac{1}{2}m_u u_i^2 + \frac{1}{2}m_v v_i^2 = \frac{1}{2}m_u u_f^2 + \frac{1}{2}m_v v_f^2$ and CoM gives us $m_u\vec{u}_i + m_v\vec{v}_i = m_u\vec{u}_f^2 + m_v\vec{v}_f^2$.
There is a special case, where one particle's initial speed is zero. Then, there are two cases when they hit:
Either, they hit along the line through their CM's. If the two balls have the same mass, the moving ball stops and the still ball takes over its momentum and starts moving in the same direction.
Or, they hit at an angle. Then, the still ball starts moving in the direction of the impulse and the moving ball changes direction to a path perpendicular to the impulse. The direction of the impulse is along the vector between the two CM's at the moment of collision.

8.2 Rigid Bodies

A rigid body is a body where the distance between two points $|\vec{r}_i - \vec{r}_j|$ is fixed. Earlier, we split the kinetic energy into a part due to motion of the CM and a part due to motion around the CM. Now, we split the motion of the rigid body into a part due to translation of the CM and a part due to motion around the CM.

8.2.1 Rotation

The **angular velocity vector** is $\vec{\omega} = \omega\hat{n}$, where $\hat{n}$ is the unit vector pointing along the axis of rotation. (Use the right hand rule to find the direction.) For a rigid body with no torque acting on it, $\vec{\omega}$ is constant. The angular velocity α is equal to $\frac{a}{r}$, with a the acceleration.

$\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$ and $|\vec{v}| = \omega r sin\phi$, where ϕ is the angle of precession.

The kinetic energy for pure rotation is (using $r\text{sin}\phi = r_\perp$):
$\frac{1}{2}\sum m_i \omega^2 r_\perp^2$

The **moment of inertia** is also denoted by I, but it is given by $I = \sum r_{\perp i}^2 m_i$, where r_{perp} is perpendicular distance from the axis of rotation. Here mass represents reluctance to move, I represents reluctance to rotate.

Then, for a rigid body, $T = \frac{1}{2} I\omega^2$ with $I = \int r_\perp^2 dm$. Note that this is the moment of inertia taken as a sum over m (an integral is a sum).

Using $\dot{\vec{r}} = \vec{\omega} \times \vec{r}$, we see $\vec{L} = \sum_i m_i(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \rightarrow \vec{L} \cdot \hat{n} = I\omega$ where $\hat{n}$ is the unit vector parallel to $\vec{\omega}$. Then, $\vec{L} = I\omega$. There are some instances where this is not the case, but they are not discussed in this course. **Special case:**
If $\vec{\tau} \parallel \hat{n}$, then $\tau = I\dot{\omega}$, where $\dot{\omega}$ is the angular acceleration, also denoted by α.
Otherwise:
$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{dL}{dt}\hat{n} + L\frac{d\hat{n}}{dt}$.

9 Rigid Body Motion, cont.

9.1 Precession

Precession is when an object rotates around some "pole" (axis of rotation). The rotational speed is denoted Ω.
$\vec{\tau} = \vec{\Omega} \times \vec{L}$ and $\Omega = \frac{Mgr}{I\omega}$.

Finding the moment of inertia for an object with uniform density:

- 1D: $\lambda dx = dm$, $M = \lambda L$, so $I = \int r_\perp^2 dm = \int r_\perp^2 \lambda dx = \int r_\perp^2 \frac{M}{L} dx$
- 2D: $\sigma dA = dm$, $M = \sigma A$
- 3D: $\rho dV = dm$, $M = \rho V$

To find the moment of inertia around some axis not through the center of mass, we use the **Parallel Axis Theorem**: For a body with mass M and I_{CM} already calculated about an axis which passes through the CM, $I_{\hat{n}} = I_{\text{CM}} + Mh^2$ where $\hat{n}$ is an axis passing through the point we are investigating that is parallel to the axis used for I_{CM}. h is the distance between the CM and said unit vector. Reason to use this: it's often easier to find the moment of inertia around the center of mass since we have the above formulae.

10 Non-Inertial Frames: Introduction and Centrifugal Force

We consider two frames with a common origin: S with (x, y, z) and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and S' with (x', y', z') and $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$. The unit vectors $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ are rotating with $\vec{\omega}$.

- If a particle is at $\vec{r}(t)$ in S, its position in S' is $\sum_{i=1}^{3} r'_i \hat{e}'_i$.
- If velocity in the S frame is $\dot{\vec{r}} = \dot{r}_i \hat{e}_i$, its velocity in the S' frame is $\dot{r}'_i \hat{e}'_i(t)$.
- $\vec{a}_S = \vec{a}_{S'} + 2\omega \times (\frac{dr}{dt})_{S'} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

Using Newton's second law, we see $m\vec{a}_{S'} = \vec{F} - 2m\vec{\omega} \times (\frac{dr}{dt})_{S'} - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$.

The last three of those are fictitious forces: the coriolis force, the Euler force and the centrifugal force. They are called fictitious because they are not the result of any physical interaction. The coriolis force only exists when the particle moves with respect to the rotating object and the Euler force only exists when the angular velocity changes.

10.1 Centrifugal force

The centrifugal force is conservative with $\vec{F}_{\text{CENT}} = -\vec{\nabla} V$, where $V = -\frac{1}{2}|\vec{\omega} \times \vec{r}|^2$. The greater r, the smaller V, so particles will fly out from the axis of rotation to lower V. In other words, the centrifugal force always points outward and perperndicular to the axis of rotation.

On Earth, we experience centrifugal force. Because of this, the gravity we feel does not point straight down toward Earth's center of mass, but slightly toward the equator.

11 Non-Inertial Frames: Coriolis Force

The corolios force is given by $\vec{F}_{\text{COR}} = -2m\vec{w} \times \vec{v}$, where $\vec{v}$ is the velocity of the object as measured in the rotating frame and $\vec{w}$ is the angular momentum vector of the rotating frame.

To find the direction of deviation, we decompose $\vec{w}$ into a component pointing straight up from the surface and one horizontal to the surface. The resulting force then points in the opposite direction of the cross product $\vec{w}_{\text{ver}}$ and $\vec{v}$.
Using angular momentum: if we view the Earth-ball system from outside, the angular momentum of the ball as it moves has to stay constant. $L = mr_{\perp}^2\omega_E$. This should stay constant, so if $r_{\perp}$ changes, it tells us that ω_E needs to change to make up for it.

To find the magnitude of deviation, we choose a convenient coordinate system where $\hat{z}$ points straight up from the surface, $\hat{x}$ points in the direction of horizontal movement, and $\hat{y}$ point so that the RHR is obeyed ($\hat{x} \times \hat{y} = \hat{z}$). Then we compute the cross product as usual to find the coriolis force. This force is also a vector $m(a_x, a_y, a_z)$. We can solve these equations if we know the initial conditions.

Part 2: Special Relativity

12 Introduction to Special Relativity

12.1 Postulates of Special Relativity:

1. Principle of Relativity: The laws of physics are the same in all inertial frames.

2. The speed of light is the same in all inertial frames.

12.2 Lorentz transformations

Galilean transformations don't work when things are travelling at the speed of light. Observers with different speeds don't agree on time intervals and distances. New transformations: Lorentz transformations:
For inertial frames S and S', where S' is moving at speed v with respect to S in the positive x direction:

- The Lorentz factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
- $t' = \gamma(t - \frac{xv}{c^2})$
- $x' = \gamma(x - vt)$
- $y' = y$
- $z' = z$
- $\Delta t' = \gamma(\Delta t - \frac{v\Delta x}{c^2})$
- $\Delta x' = \gamma(\Delta x - v\Delta t)$
- $\Delta t = \gamma(\Delta t' + \frac{v\Delta x'}{c^2}$
- $\Delta x = \gamma(\Delta x' + v\Delta t'$

12.3 Spacetime Diagrams:

A spacetime diagram has ct as the y axis and x as the x axis. This means both have units of length.

We use spacetime diagrams to picture the difference between frames. For a frame that is moving with speed v, the time axis has slope $\frac{c}{v}$ and the distance axis has slope $\frac{v}{c}$.
Lines parallel to x' are lines of simultaneity, and things along that line happen at exactly the same time.

Some terms:

- An event is anything that happens at a specific time and place and can be plotted as a point on the spacediagram.
- A worldline is the curve traced out by an object or observer. If each second in that object's life were an event, all the seconds together would create the curve.

13 Observers (spacetime diagrams continued)

13.1 Simultaeneity and causality

Two events that occur at the same time in an inertial frame are called **simultaneous**. They may not be simultaneous from the point of view of another inertial frame moving at a different speed (see Lorentz transformations for intervals). Two frames moving in opposite directions can even see events in a different order from each other.

One event A can cause another event B, but only if light can travel between the two in time. For any event, there is a light cone, which consists of the the lines with slopes 1 and -1 intersecting at the event. If B is on or in A's light cone and above A in the spacetime diagram (the future light cone), A could have caused B. If B is on or in A's light cone under A (the past light cone), B could have caused A. If B is outside of A's light cone, there is no physical way the two could have been connected.

13.2 Time and length dilation

Time and length measured in one frame do not match those measured in another frame. The proper time and proper length are those measured by the object itself (assuming the object moves at uniform velocity). More officially, the proper time and length are those as measured in the frame in which the object is at rest, which is in fact its own frame. To find the time and length in other frames, we use:
$T^{\text{other}} = \gamma T^{\text{proper}}$ and $L^{\text{other}} = \frac{L^{\text{proper}}}{\gamma}$

14 Spacetime

14.1 Doppler Effect

If two objects A and B are moving relative to each other, and B sends signals to A with a certain frequency, A does not actually receive these signals at that frequency. To find the time A observes, we use:
$T^{\text{obs}} = \gamma T^{\text{emit}}$. This effect is called time dilation. An important distinction, however, is that T^{obs} is not the time period between flashes received by A. Observations are done by an "omniscient" observer who knows what's going on everywhere when it happens. So they don't actually see the light reach them, they just know that it's being sent. For a normal person, they see the flashes with interval
$T^{\text{rec}} = \sqrt{\frac{c+v}{c-v}} T^{\text{emit}}$. Since $f = \frac{1}{T}, f^{\text{rec}} = \sqrt{\frac{c-v}{c+v}} f^{\text{emit}}$. And since the wavelength $\lambda = ct, \lambda^{\text{rec}} = \sqrt{\frac{c+v}{c-v}} \lambda^{\text{emit}}$. This last equation is why, if B moves away from A, A sees the light in a redshifted way, and if B moves toward A, A sees the light blueshifted.

The above description is the difference between what we mean when we say someone observes something (omniscience) and when someone sees something (normal person using their eyes).

14.2 Time addition

In Newtonian physics, if we see a car move away from us with speed v and the car sees an object moving away from it with speed u, we can add the two speeds and conclude that the object is moving away from us at speed $v + u$. In special relativity, this doesn't work. We need to use the conversion: $u = \frac{u'+v}{1+\frac{vu'}{c^2}}$ where v is the speed of the car and u' is the speed of the object as seen from the car, and u is the speed we see the object as having. The inverse relation is
$u' = \frac{u-v}{1-\frac{uv}{c^2}}$.

14.3 Spacetime

In special relativity, neither time nor space are absolute. We do have an invariant value, which is the spacetime interval Δs^2, defined as:
$\Delta s^2 := c^2 \Delta t^2 - \Delta x^2$. For any two events, this value is the same regardles of frame.
The spacetime interval can tell us about the causality of two events:

- If $\Delta s^2 > 0$ the events are timelike separated and are inside each other's lightcone.
- If $\Delta s^2 = 0$ the events are lightlike separated and are on each other's lightcone.

- If $\Delta s^2 < 0$ the events are spacelike separated and outside of each other's lightcone.

The spacetime interval can also be expressed using 4-vectors: $\Delta s^2 = \Delta X^T \eta \Delta X$ where $X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ and η is the Minkowski metric $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

15 Dynamics

Dynamics is about the basic elements of motion: velocity, acceleration and momentum. Our goal is to show that, the laws of physics are compatible with the postulates that (1) physics should be the same in all inertial frames and (2) there is a universal speed limit.

15.1 New parametrisation

In Newtonian mechanics, we parametrised things using t. However, in special relativity, different frames don't agree on t. If we use paramter τ, all frames will agree on the following 4-vectors since τ is invariant between frames.

- $\Delta\tau = \frac{\Delta s}{c}$
- $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma$ with u the speed of the particle
- $T = \int d\tau = \int \frac{dt}{\gamma}$, where T is the total time passed in S'
- $X(\tau) = \begin{pmatrix} ct(\tau) \\ x(\tau) \\ y(\tau) \\ z(\tau) \end{pmatrix} = \begin{pmatrix} ct(\tau) \\ \vec{x}(\tau) \end{pmatrix}$. This is called a 4-vector: it's like a normal vector but with a time component as the first element. More officially, a 4-vector is a vector that transforms under linear transformations: $X' = \Lambda X$, where Λ is the Lorentz matrix, which is the matrix that transforms Lorentz transformations.
- 4- velocity is $U = \frac{dX}{d\tau} = \gamma \begin{pmatrix} c \\ \vec{u} \end{pmatrix}$
- 4 acceleration is $A = \frac{dU}{d\tau}$
- A and U are 4-vectors, so they transform with the Lorentz matrix.
- $U \cdot U = c^2$, so while U may transform, its dot product $U \cdot U = |U|^2$ does not.
  $P \cdot P = m^2c^2$ so this is also invariant.
  In general, the inner product of a 4-vector with itself is invariant.
- 4-momentum is $P = mU = \begin{pmatrix} m\gamma(u)c \\ \vec{p}. \end{pmatrix}$ where $\gamma(u)$ is gamma as a function of the velocity of the object. The first component (time component) of the 4-momentum is equal to the Taylor expansion $\frac{1}{c}(mc^2 + \frac{1}{2}mu^2 + \frac{3}{8}(\frac{u}{c})^4 + \frac{5}{16}(\frac{u}{c})^6 + ...$ where the first term does not depend on velocity and is called the rest energy of the object, the second term is called the non-relativistic kinetic energy, and the terms after that are relativistic corrections.
  Finally, this time component is equal to E/c. We see that as the speed of

the object approaches the speed of light, the energy approaches infinity, which means reaching that speed requires infinite energy. So no particle with mass can reach the speed of light.

- The energy of an object is equal to $E = \sqrt{m^2c^4 + \vec{p}^2c^2}$. When the speed is much smaller than c, this goes to $mc^2 + \frac{\vec{p}^2}{2m}$, which is the sum of the rest energy and the kinetic energy.

15.2 Laws of Physics using the new parametrisation

Conservation of momentum:
The 4-momentum can be added like a normal vector. Then, in an isolated system, it holds that the sum of all initial 4-momenta is equal to the sum of all final 4-momenta.

Conservation of energy:
Mass and energy are not separately separated, but in an isolated system the total of the initial time components of momentum does equal the total of the final time components. Sometimes this means that mass turns to energy or energy turns to mass.

Special case: light
Light has no mass, but it does have momentum.
$P_{\text{light}} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$
The energy of light is $pc = hf$ where h is Planck's constant and f is the frequency.

4-force F:
$\frac{dP}{dt} = F = \begin{pmatrix} \text{time component of F} \\ \vec{F} \end{pmatrix} = mA$